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Ergodicity Economics in Plain English

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ABSTRACT

Ergodicity economics (EE) applies a modern mathematical formalization to familiar financial concepts to reveal implications and consequences that were previously unseen. EE quantifies the differences and the trade-offs between the collective meaning and the individual meaning of financial methods. EE's perspective opens up previously unseen distinctions for evidence-based recommendations. These distinctions enable the creation of previously unavailable recommendations for the explicit benefit of individual clients. This differentiating impact on economic theory, asset valuation, product development, and advisory best practices is developing rapidly.

INTRODUCTION

Ergodicity economics (EE) is a mathematical model of growth with historical anchors that go back to Euclid (third century BCE) in Book V, Proposition 25: *Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν*. "If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes)." A special case occurs when the middle terms are the same, in which case this proposition gives us a corollary: The arithmetic mean of two magnitudes is greater than their geometric mean. This inequality of means has been known for a long time.

In the late 2000s, a physicist affiliated with the Imperial College in London, the University of California, Los Angeles, and the Santa Fe Institute, Ole Peters, turned his gaze from weather patterns to wealth patterns and wrote papers that provided clear distinctions between:

- Methods of averaging (arithmetic means versus geometric means)
- Meanings of averaging (ensemble expectation versus time average)
- Reasons for these different meanings (additive versus multiplicative growth dynamics)

These are distinctions with a difference because the average experience of an ensemble over many trajectories may not be the average experience of an individual over a single life history. Using ensemble expectations inappropriately, i.e., for non-ergodic observables, misleads individuals because it implies a physical system of counterfactuals that cannot exist in a single life trajectory.

Ergodicity economics (EE) is an important extension of traditional finance. It solves the limitations of the assumption of ergodicity that is at the core of economics since 1947.

The assumption of ergodicity implies a clockwork sameness between the past and the future. Ergodic systems have no history and thus can have an equilibrium. However, any system that grows is non-ergodic because it has a history generated by its unique growth rates.

The terminology of EE will feel unfamiliar. It was brought to us by physicists trained in the German tradition of mathematics. At the very least, the mathematical notation requires a period of adaptation. Further, EE's terminology can be a confusing blend of old and new concepts, thus the addition of a glossary at the end of this paper.

The context for EE also will feel unfamiliar. It pushes us to acknowledge the presence, the meaning, and the importance of other types of means, above and beyond the familiar arithmetic average.

The value that it brings to the financial industry is well worth the familiarization effort. As a counterweight to the age of holistic financial planning, EE provides a quantitative approach for direct product selection by matching clients to investment vehicles with metrics that include growth dynamics, time-averages, minimum holding periods, and maximum investment allocations.

LITERATURE REVIEW

EE applies a modern mathematical formalization to familiar concepts to reveal relationships and consequences that previously were unseen. This combination of old and new changes our understanding of many topics, and finance in particular, in fundamental ways, akin to seeing reality through a new lens.

In 2011, Peters published two papers that started the formal development of ergodicity economics: "Optimal Leverage from Non-ergodicity" (Peters 2011a) and "The Time Resolution of the St. Petersburg Paradox" (Peters 2011b). In 2013, a preprint submission to the *Journal of Economic Theory*, "Stochastic Market Efficiency," showed his affiliation with the London Mathematical Laboratory (LML). Since then, he has published more than ten papers exploring the many financial puzzles, paradoxes, and anomalies that are resolved by EE. His co-authors include Alexander Adamou, Yonatan Berman, Murray Gell-Mann, Mark Kirstein, William Klein, Diomedes Mavroyiannis, and Maximilian J. Werner.

In a series of tweets in 2020, Ole Peters expanded EE's intellectual lineage to include Gerolamo Cardano, Ludwig Boltzmann, William Allen Whitworth, Kiyosi Ito, J. L. Kelly, Edward O. Thorpe, Anthony B. Atkinson, Thomas Cover, Robert Fernholz, A. Kacelnick, Matteo Marsili et al., Jean-Philippe Bouchaud et al., and Menahem E. Yaari et al.

J. L. Kelly is one of the better known of these historical examples of understanding the difference between arithmetic and geometric means. EE frames up the general case. Kelly and others are special cases. The Kelly Formula is a special case where the growth dynamic is multiplicative and the transformation function is the logarithm.

This list of EE-related authors is impressive, but it is remarkably small compared to the number of authors of finance papers who do not make use of this understanding of the differentiated meaning between ensemble expectation and time average. This may change over time because EE is a research framework that encompasses and extends the body of knowledge of traditional finance.

Non-ergodic. An observable is non-ergodic when its time average is not equal to its ensemble average at the ensemble and time limits of infinity. This situation arises in open systems where individual outcomes can differ in magnitude and in sign from the ensemble outcome due to path dependency for individual trajectories. This is a historical understanding of time where "symmetry breaking" between the past and the future is a creative force. Under such circumstances, the arithmetic mean (ensemble average) overstates the geometric mean (time average) and using insufficiently constrained utility functions results in willful blindness to excessive risk.

Ergodic. On the other hand, an observable is ergodic when its time average is equal to its ensemble average at the ensemble and time limits of infinity. This is a special situation that arises in closed systems viewed by an external observer, and with no dependence on initial conditions, is thus able to reach a long-term equilibrium. This is an analytic understanding of time that "integrates time away" where the future is a symmetrical mirror of the past. Under such circumstances, the arithmetic mean (ensemble average) is a reasonable proxy and calculation simplification for the behavior of the geometric mean (time average).

To explore the behavior of ergodic and non-ergodic observables beyond the contingency of historical data requires generative processes for discrete as well as continuous random variables. EE uses several types of generative processes, including:

- Geometric Brownian motion that connects EE to portfolio construction (Peters and Klein 2013)
- Gambles (multiplicative binomial processes) that connect EE to decision theory (Peters and Gell-Mann 2016)
- Reallocating geometric Brownian motion that connects EE to economics (Berman et al. 2020)

Given these inputs of noisy choices, EE models spin the following three mathematical threads into its new decision-making cloth:

MODES OF REPETITION

- Growth dynamics are mathematical expressions (differential equations) that calculate trajectories (stochastic processes that represent life histories of wealth, distributions of returns, etc.).
- Additive growth dynamics are akin to receiving a monthly income.
- Multiplicative growth dynamics are akin to earning interest on a bank account.

SUMMARY STATISTICS

- Means are statistics, summaries of the characteristics of trajectories, and distributions as a single number (scalar) between two numbers (maximum and minimum). Among the many types of mathematical means that exist (including the contraharmonic, quadratic, arithmetic, median, logarithmic, geometric, and harmonic means), EE focuses on the meanings of the arithmetic mean and the geometric mean.
- The meaning of the arithmetic mean, adding up observations and dividing them by the number of observations, is the expectation of the ensemble across its possible realizations for a specific time frame. This is an averaging across parallel universes.
- The meaning of the geometric mean, multiplying observations and taking their n^{th} root based on the number of observations, is the average of an individual trajectory over the long term. This is a realization over a single trajectory.

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ERGODIC TRANSFORMATIONS

- EE uses specific mathematical transformations to modify the characteristics of non-ergodic trajectories and distributions into ergodic observables.
- Ergodic transformations connect EE to expected utility theory (Peters and Adamou 2018).
- The logarithmic transformation holds a special place because it is both a transformation of periodic return multipliers into growth rates and an ergodic transformation of non-ergodic trajectories with multiplicative growth dynamics.

Finally, EE creates testable predictions for its own theory and results, including the internal decision criterion of growth optimality: Are people first-order growth-optimizers based on the circumstances of their growth dynamic? Early results, such as Meder et al. (2019), validate EE's prediction of growth optimality. Peters and his colleagues at the London Mathematical Laboratory are actively encouraging additional empirical tests.

IMPLICATIONS

Kirstein (2015) documents how ergodicity was formally created by Ludwig Boltzmann in 1884 to solve a calculation problem in modeling the behavior of gases in a closed jar. He was explicit that this equalization of arithmetic mean and geometric mean was a mathematical trick ("*kunstgriff*") to simplify calculations.

Boltzmann's simplification was introduced into economics by Paul Samuelson in 1947 (*Foundations of Economic Analysis*). This explicit introduction seemed tacitly accepted without objections, long forgotten by all involved until Peters showed the implications and consequences of its meaning on the open systems of economics and finance.

Samuelson's ergodic axiom always answers "yes" to the question: Is the time average of an observable equal to its expectation value? This introduction from closed-system physics to open-system economics, the ergodic axiom, is the foundation for many puzzles, paradoxes, and anomalies that arise between academic theory and empirical observations in finance and economics.

The assumption of the ergodic axiom in open systems leads to the ergodic fallacy: The false belief in relationships that do not exist or that change constantly. EE does not assume the ergodic axiom. Instead, it calculates the explicit conditions where ergodicity may, or may not, apply. Thus, EE resolves many puzzles, paradoxes, and anomalies in traditional finance, including:

St. Petersburg Paradox (Peters 2011b). Unlike the time average solution, the ensemble solution is dominated by the infinite payout

Equity Premium Puzzle (Peters and Adamou 2013). Normal fluctuation/error rate of a noisy growth process

The Puzzle of Cooperation (Peters and Adamou 2015). Variance reduction explains the paradox without recourse to external explanations

The Puzzle of Inequality (Adamou and Peters 2016). High variance leads to winner-takes-all growth where the ensemble average is a gain, but the time average is a loss for nearly all in the ensemble

The Paradox of Insurance (Peters and Adamou 2017). Price ranges where both parties can increase their respective time average growth rates

Circular Axiomatics (Peters and Adamou 2018). From unconstrained psychological re-weighting of wealth to non-linear transformations that define ergodic observables

Preference Reversals (Adamou et al. 2019; Berman and Kirstein 2020). Time average changes over changing time horizons and changing payouts

Probability Weighting (Peters et al. 2020). At a minimum, differences in variance assumptions between the disinterested observer's model and the decision-maker's model

EE provides a first-order, internally generated, growth-based decision criterion for recommendations, including an error range, and testable predictions. It opens up a vast field for additional research both within and outside of EE models. For instance, if first-order preferences are determined by growth dynamics, are there second-order personality differences whose influence can be disambiguated from the first-order preference?

CONSEQUENCES

EE sees reality through the lens of growth rates, a world view where we live on the expanding edge of an exploding cosmos, evolving organic life, accumulating knowledge, developing technologies, and complex economies. It is a set of principles, derived from physics and applied to economics, to improve our understanding of mathematical averages from mechanical methodologies to meaningful interpretations. Specifically, the meaning of the arithmetic mean (expected value, expectation, ensemble average) is averaging over counterfactuals, i.e., parallel universes. The meaning of the geometric mean (time average) is averaging over the long term of an individual trajectory.

GROWTH OPTIMALITY

An observable with a multiplicative growth dynamic can have a positive arithmetic mean and a negative geometric mean. This means that the ensemble expectation is a gain but the individual realization is a loss, i.e., the market shows positive returns but most investors lose money. This is caused by the oversized influence of a small number of winner-take-all investors. EE's geometric mean formula shows that the driver for such disparities is the volatility (variance) of the growth rate.

EE provides a new default model to anticipate the behavior of individuals: growth optimality. This new default model posits that people are first-order growth optimizers based on the circumstances of their growth dynamics. Growth optimality is based on making the distinction and calculating the differences between maximizing the expectation value of ergodic observables versus maximizing the expectation value of non-ergodic observables.

CLIENT BEHAVIOR

Insights derived from EE's solutions to traditional puzzles, paradoxes, and anomalies show that for individuals seeing decisions through the lens of an additive growth dynamic, i.e., no skin in the game, no bankruptcy, and expected monthly payments, the growth optimal strategy is to lever up exposure to risky assets because there are no "game-over" consequences from zero. Is your decision-making environment an additive growth dynamic?

On the other hand, for individuals seeing decisions through the lens of a multiplicative dynamic, i.e., skin in the game, threat of bankruptcy, and uncertain returns on investment, the growth optimal strategy is cautious exposures to risky assets because there is no recovering from zero. Is your decision-making environment a multiplicative growth dynamic?

ADVISOR BEHAVIOR

Seeing a client's multiplicative growth dynamic through the lens of an advisor's additive growth dynamic, i.e., optimizing a multiplicative dynamic with a non-ergodic arithmetic mean, leads to over-aggressive expectations. Similarly, seeing an additive growth dynamic through the lens of a multiplicative dynamic, i.e., optimizing an additive dynamic with a geometric mean, leads to overcautious expectations.

Growth creates noisy patterns that confound commonly used metrics such as expectation values. EE makes the confounding observable explicit (growth dynamic), flags false (non-ergodic) expectations, and provides first-order (time average) recommendations that do not require an appeal to external, unconstrained (preferencing) variables.

PRODUCT SELECTION

EE extends our understanding upstream as well as downstream from the means of non-ergodic observables. Growth dynamics are upstream from the means of non-ergodic observables and quantify risk tolerance without recourse to self-referential psychological explanations. Ergodic transformations are downstream from the means of non-ergodic observables and quantify utility theory without recourse to self-referential psychological explanations.

Recognizing the meaning of the means makes it possible to provide first-order optimization, asset allocation, and product selection recommendations to clients without a recourse to external and unreliable inputs such as psychological preferences.

OPEN QUESTIONS

EE's perspective lifts us above traditional academic and explanatory silos to combine old and new into improved best practices. For instance:

- What behavioral finance observations could be confounded variables caused by the absence of an explicit specification of the relevant growth dynamic?
- What phases of investment planning and retirement planning have additive versus multiplicative growth dynamics?
- When does it make sense to shift from risk-seeking to risk-avoiding exposures, not because of one's changing psychological preferences but because of one's changing growth dynamic?
- What do you use, currently, to make these decisions?

CONCLUSIONS

EE is a new perspective, expressed as a mathematical language that arises from calculating ergodic observables for non-ergodic growth processes. EE works alongside the traditional economic models to transcend their results. In particular, EE is able to provide recommendations that are organic to the model and do not require external assumptions based on psychology.

EE quantifies the differences and the trade-offs between the collective meaning and the individual meaning of financial methods. EE's perspective opens up previously unseen distinctions for evidence-based recommendations. These distinctions enable the creation of previously unavailable recommendations for the explicit benefit of individual clients. This differentiating impact on economic theory, asset valuation, product development, and advisory best practices is developing rapidly.

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GLOSSARY

EE's terminology can be a confusing blend of old and new concepts, thus the addition of a glossary to this paper.

Arithmetic mean

Method: The arithmetic mean sums up observations and divides them by the number of observations.

Ensemble average

Method: The ensemble average takes the arithmetic mean at the ensemble limit of infinity.

Meaning: The ensemble average is an average across the ensemble of possible realizations (trajectories), thus an average across counterfactuals, or parallel universes.

Ergodic observable

An observable whose ensemble average is equal to its time average.

Ergodic transformation

An ergodic transformation is a functional transformation of a trajectory that enables the computation of ergodic observables.

Gamble

A gamble is a random variable that generates a set of possible changes in monetary wealth with associated probabilities over a number of elementary events. The gamble, a discrete version of geometric Brownian motion, is the first data generation mechanism that readers encounter in Ole Peters' Course Notes.

Geometric mean

Method: The geometric mean multiplies observations and takes the n^{th} root.

Growth dynamic

A growth dynamic is a differential equation that defines the decision-maker's perspective used to choose between gambles. It is the mode of repetition that uses a gamble's specific inputs to create a trajectory, a stochastic process.

Observable

An observable can be a time series, a trajectory of prices, that forms the basis for other observables such as periodic returns, multipliers, and growth rates. Observables can be ergodic or non-ergodic.

Time average

Method: The time average takes the geometric mean at the time limit of infinity.

Meaning: The time average is an average across the time line of a single realization, or trajectory.

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